Advanced Automatic Control MDP 444

Lecture 2

Staff boarder

Prof. Dr. Mostafa Zaki Zahran

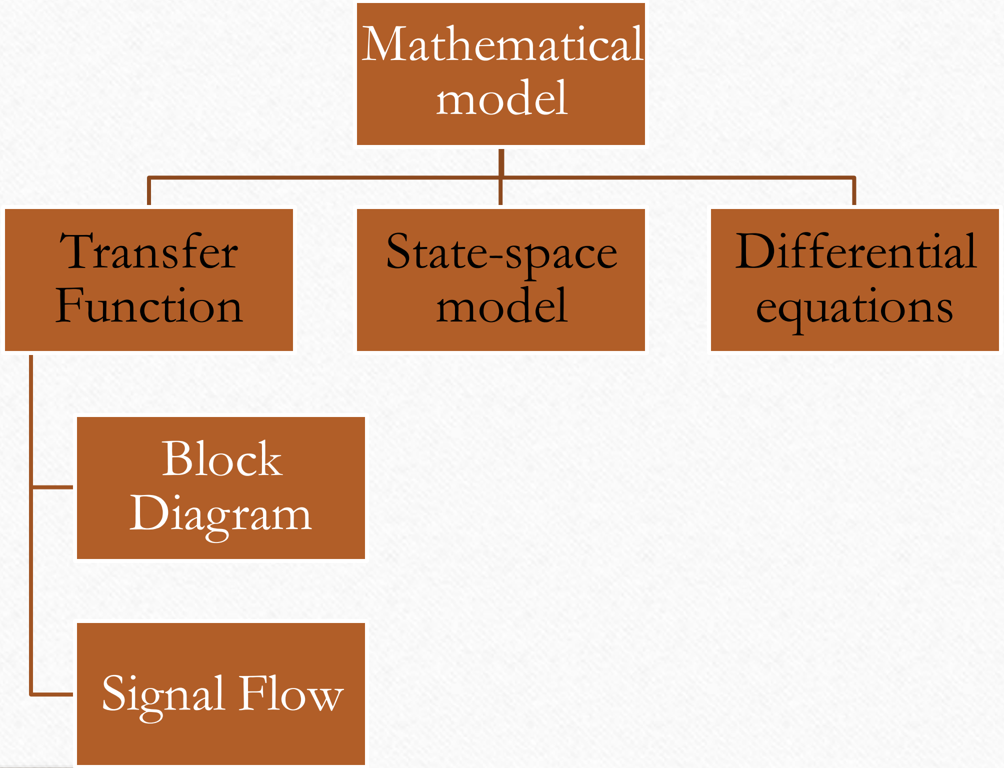
Dr. Mostafa Elsayed Abdelmonem

Advanced Automatic Control
MDP 444

- **Lecture aims:**
  - Recognize that state variable models can describe the dynamic behavior of physical systems and can be represented by block diagrams and signal flow graphs.
  - Understand the important role of state variable modeling in control system design

Mathematical Modeling

- **Transfer Function**



Signal Flow

A **signal-flow graph** is a diagram consisting of nodes that are connected by several directed branches and is a graphical representation of a set of linear relations.

The basic element of a signal-flow graph is a unidirectional path segment called a **branch**

A **loop** is a closed path that originates and terminates on the same node. Two loops are said to be **nontouching** if they do not have a common node

$$T_{ij} = \frac{\sum_k P_{ijk} \Delta_{ijk}}{\Delta},$$

P_{ijk} = gain of kth path from variable x_i to variable x_j,

Δ = determinant of the graph,

Δ_{ijk} = cofactor of the path P_{ijk},

$$\Delta = 1 - \sum_{n=1}^{N} L_n + \sum_{\substack{n,\,m \\ \text{nontouching}}} L_n L_m - \sum_{\substack{n,\,m,\,p \\ \text{nontouching}}} L_n L_m L_p + \cdots$$

$\Delta = 1 -$ (sum of all different loop gains)

$+$ (sum of the gain products of all combinations of two nontouching loops)

$-$ (sum of the gain products of all combinations of three nontouching loops)

$+ \cdots$.

The cofactor Δ_{ijk} is the determinant with the loops touching the kth path removed.

Signal Flow

The paths connecting the input *R(s)* and output *Y(s)* are

$$P_1 = G_1G_2G_3G_4 \text{ (path 1)} \quad \text{and} \quad P_2 = G_5G_6G_7G_8 \text{ (path 2)}$$

There are four self-loops:

$$L_1 = G_2H_2, \quad L_2 = H_3G_3, \quad L_3 = G_6H_6, \quad \text{and} \quad L_4 = G_7H_7$$

Loops *L1* and L2 do not touch L3 and L4. Therefore, the determinant is

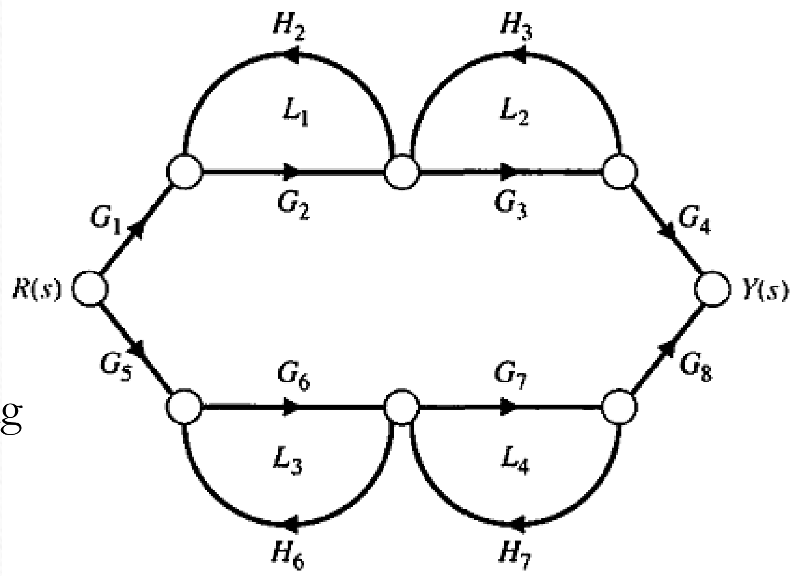
$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4) + (L_1L_3 + L_1L_4 + L_2L_3 + L_2L_4)$$

The cofactor of the determinant along path 1 is evaluated by removing the loops that touch path 1 from Δ. $\Delta_1 = 1 - (L_3 + L_4)$

Similarly, the cofactor for path 2 is $\Delta_2 = 1 - (L_1 + L_2)$

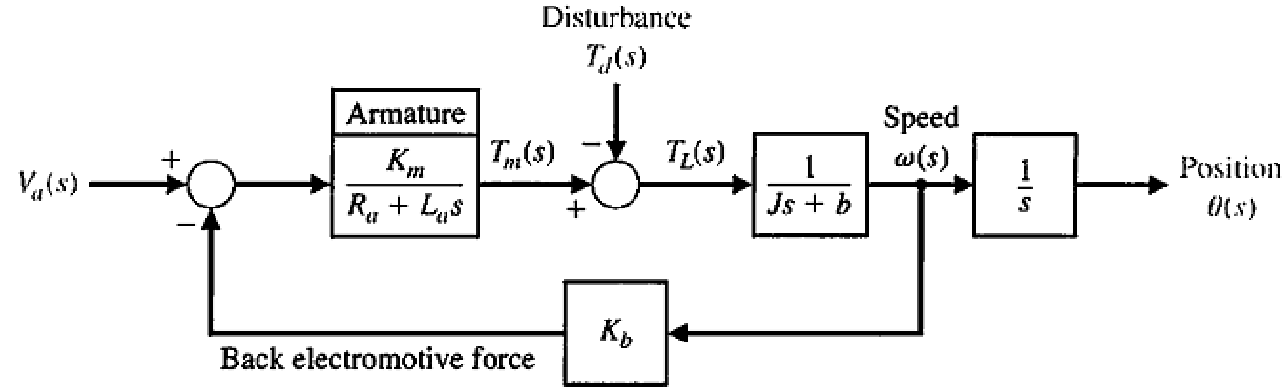
Therefore, the transfer function of the system is

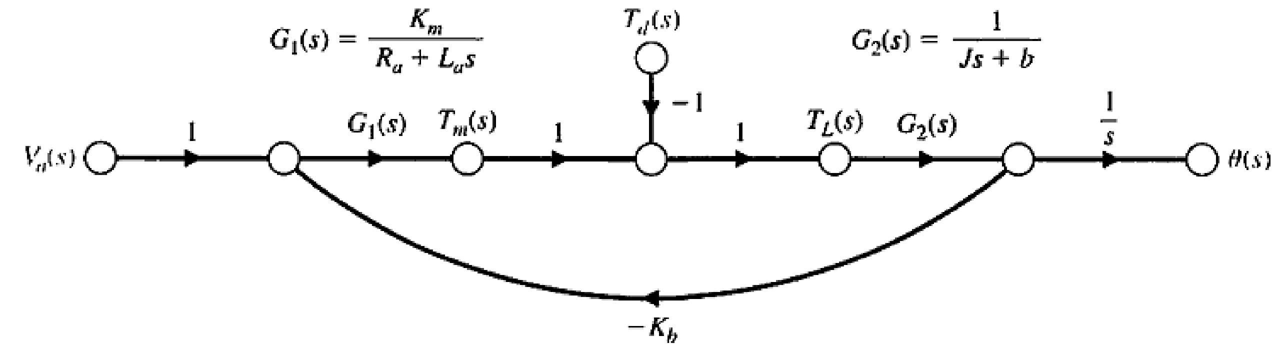
$$\frac{Y(s)}{R(s)} = T(s) = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_1G_2G_3G_4(1 - L_3 - L_4) + G_5G_6G_7G_8(1 - L_1 - L_2)}{1 - L_1 - L_2 - L_3 - L_4 + L_1L_3 + L_1L_4 + L_2L_3 + L_2L_4}$$



Signal Flow

The armature-controlled DC motor



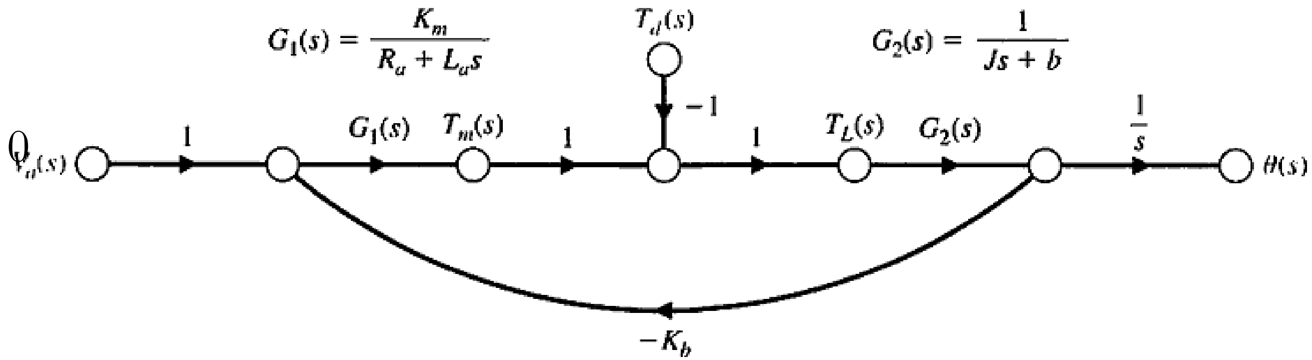


Signal Flow

The armature-controlled DC motor

Using Mason's signal-flow gain formula, transfer function for $\theta(s)/Va(s)$ with $Td(s) = 0$

The forward path is $P1(s)$, which touches the one loop, $L1(s)$, where



$$P_1(s) = \frac{1}{s}G_1(s)G_2(s) \quad \text{and} \quad L_1(s) = -K_bG_1(s)G_2(s).$$

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4) + (L_1L_3 + L_1L_4 + L_2L_3 + L_2L_4)$$

$$T(s) = \frac{P_1(s)}{1 - L_1(s)} = \frac{(1/s)G_1(s)G_2(s)}{1 + K_bG_1(s)G_2(s)} = \frac{K_m}{s[(R_a + L_as)(Js + b) + K_bK_m]},$$

State Space Equations

- **State equations** is a description which relates the following four elements: input, system, state variables, and output

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

Matrix A has dimensions nxn and it is called the **system** matrix, having the general form

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Matrix B has dimensions nxm and it is called the **input** matrix, having the general form

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix}$$

Matrix C has dimensions pxn and it is called the **output** matrix, having the general form

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{bmatrix}$$

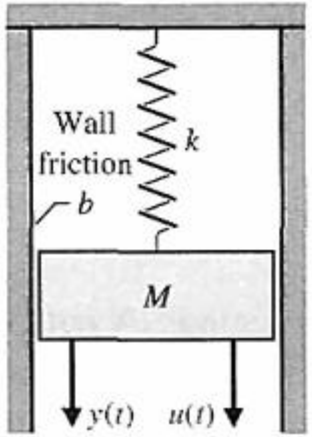
Matrix D has dimensions pxm and it is called the **feedforward** matrix, having the general form

$$\mathbf{D} = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ d_{21} & d_{22} & \cdots & d_{2m} \\ \vdots & \vdots & & \vdots \\ d_{p1} & d_{p2} & \cdots & d_{pm} \end{bmatrix}$$

State Space Representation

- **The general form of a dynamic system**

The concept of a set of state variables that represent a dynamic system can be illustrated in terms of the spring-mass-damper system. A set of state variables sufficient to describe this system includes the position and the velocity of the mass.



- We will define a set of state variables as *(x1, x2),* where

$$x_1(t) = y(t) \quad \text{and} \quad x_2(t) = \frac{dy(t)}{dt}. \qquad \frac{dx_1}{dt} = x_2$$

To write Equation of motion in terms of the state variables, we substitute the state variables as already defined and obtain

$$M\frac{dx_2}{dt} + bx_2 + kx_1 = u(t) \qquad M\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = u(t)$$

Therefore, we can write the equations that describe the behavior of the spring-mass damper system as the set of two first-order differential equations

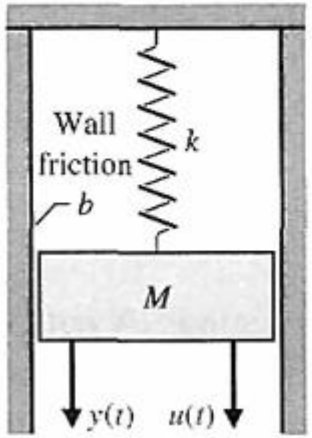
$$\frac{dx_2}{dt} = \frac{-b}{M}x_2 - \frac{k}{M}x_1 + \frac{1}{M}u$$

State Space Representation

- **State space matrix**

$$\frac{dx_1}{dt} = x_2 \qquad \frac{dx_2}{dt} = \frac{-b}{M}x_2 - \frac{k}{M}x_1 + \frac{1}{M}u$$

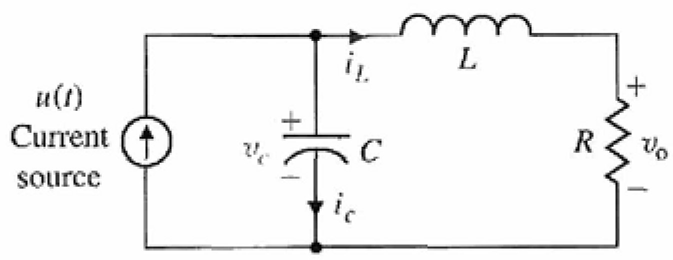
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-k}{m} & \frac{-b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} [u_{(t)}]$$



$$M\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = u(t)$$

State Space Representation

- **RLC circuit example**
- The state of this system can be described by a set of state variables *(x1, x2)*, where *x1* is the capacitor voltage *vo(t)* and *x2* is the inductor current $i_L\{t\}$.
- Utilizing Kirchhoff's current law at the junction



$$i_c = C\frac{dv_c}{dt} = +u(t) - i_L$$

OR

Kirchhoff's voltage law for the right-hand loop provides the equation describing the rate of change of inductor current as

$$L\frac{di_L}{dt} = -Ri_L + v_c$$

The output of this system is represented

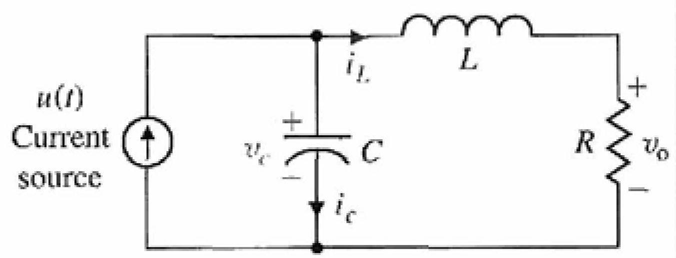
$$v_o = Ri_L(t)$$

State Space Representation

RLC circuit example

- rewrite Equations as a set of two first-order differential equations in terms of the state variables *x1* and *x2* as follows:

$$\frac{dx_1}{dt} = -\frac{1}{C}x_2 + \frac{1}{C}u(t) \qquad \frac{dx_2}{dt} = +\frac{1}{L}x_1 - \frac{R}{L}x_2$$



$$i_c = C\frac{dv_c}{dt} = +u(t) - i_L$$

$$L\frac{di_L}{dt} = -Ri_L + v_c$$

$$v_o = Ri_L(t)$$

- The output signal is then

$$y_1(t) = v_o(t) = Rx_2$$

- obtain the state variable differential equation for the *RLC*

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & \frac{-1}{C} \\ \frac{1}{L} & \frac{-R}{L} \end{bmatrix}\mathbf{x} + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix}u(t)$$

- and the output as

$$y = [0 \quad R]\mathbf{x}$$

TRANSFER FUNCTION FROM THE STATE EQUATION

- Obtain a transfer function *G(s),* Given the state variable equations. Recalling Equations :where v is the single output and *u* is the single input.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x} + \mathbf{D}u$$

  The Laplace transforms of Equations

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}U(s)$$
$$Y(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}U(s)$$

  where B is an *n* x 1 matrix, since *u* is a single input, we obtain

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}U(s)$$
$$\mathbf{X}(s) = \Phi(s)\mathbf{B}U(s)$$

- we obtain state transition Matrix

$$[s\mathbf{I} - \mathbf{A}]^{-1} = \Phi(s)$$

TRANSFER FUNCTION FROM THE STATE EQUATION

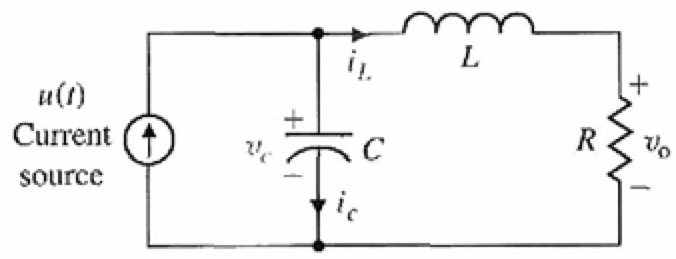
- Transfer function *G(s)*: *G(s)* = *Y(s)*/*U(s)* is

$$G(s) = \mathbf{C}\Phi(s)\mathbf{B} + \mathbf{D}$$

- Let us determine the transfer function *G(s)* = *Y(s)*/*U(s)* for the *RLC* circuit, described by the differential equations

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & \frac{-1}{C} \\ \frac{1}{L} & \frac{-R}{L} \end{bmatrix}\mathbf{x} + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix}u$$

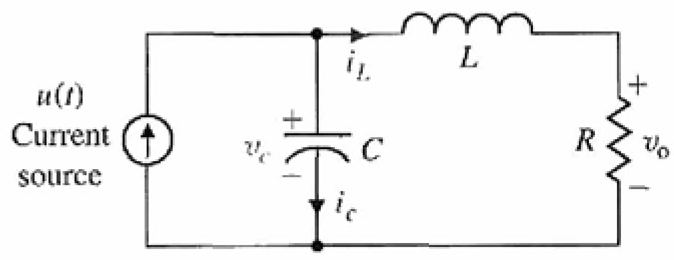
$$y = [0 \quad R]\mathbf{x}.$$



TRANSFER FUNCTION FROM THE STATE EQUATION

- Then we have

$$[s\mathbf{I} - \mathbf{A}] = \begin{bmatrix} s & \dfrac{1}{C} \\ \dfrac{-1}{L} & s + \dfrac{R}{L} \end{bmatrix}$$



- Therefore, we obtain

$$\Phi(s) = [s\mathbf{I} - \mathbf{A}]^{-1} = \frac{1}{\Delta(s)} \begin{bmatrix} \left(s + \dfrac{R}{L}\right) & \dfrac{-1}{C} \\ \dfrac{1}{L} & s \end{bmatrix}$$

$$\Delta(s) = s^2 + \frac{R}{L}s + \frac{1}{LC}$$

- Then the transfer function is

$$G(s) = [0 \quad R] \begin{bmatrix} \dfrac{s + \dfrac{R}{L}}{\Delta(s)} & \dfrac{-1}{C\Delta(s)} \\ \dfrac{1}{L\Delta(s)} & \dfrac{s}{\Delta(s)} \end{bmatrix} \begin{bmatrix} \dfrac{1}{C} \\ 0 \end{bmatrix} = \frac{R/(LC)}{\Delta(s)} = \frac{R/(LC)}{s^2 + \dfrac{R}{L}s + \dfrac{1}{LC}}$$

Model Examples

- Pulse Width Modulation (PWM)